

CALCULATION OF MAGNETIC FIELDS AND CURRENTS IN AXISYMMETRIC SYSTEMS OF INDUCTIVELY COUPLED MOVING CONDUCTORS

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UDC 537.311.5

An algorithm for calculating magnetic fields and currents in axisymmetric systems of inductively coupled moving and stationary conductors is developed using a hybrid method which combines the finite and boundary element methods. The finite element method is used to approximate the unsteady diffusion equation for the θ -component of the magnetic vector potential in the conductors, and the boundary-element method is employed to eliminate the space around the conductors. The proposed method takes into account the connections of the conductors with each other or with external energy sources by means of ideal electrical circuits with lumped parameters R , L , and C . An effective method is developed to take into account the external circuits by an appropriate modification of the mass matrix and the source vector of the obtained system of ordinary differential equations. Examples of using the method to calculate the fields of single- and multi-turn solenoids, magnetic flux concentrators, and induction accelerators with various methods of delivering external electromagnetic energy are considered. The high computational efficiency of the method is shown, in particular, for the case of constant electrothermal properties and sizes of the conductors.

Key words: magnetic field, inductor, accelerated solid.

Introduction. In many practically important cases related to the acceleration of solids to high velocities, generation of high magnetic fields etc., the determination of the force action and thermal load on conductors requires a detailed calculation of the magnetic field and current distributions in the volume of the conductors taking into account the dynamics of energy delivery to the system, which is determined by the external energy source and the parameters of the energy delivery circuits.

The electrophysical setup studied here should be described using a model with distributed parameters, whereas the analysis can be simplified by treating the external circuits as lumped-parameter circuits. This involves two problems: the need to take into account the finite size of the setup and, hence, to determine the fields in the infinite space around the system of conductors studied; the need to match the currents in the two subsystems: the spatially distributed currents in the setup and the total currents in the external electrical circuits. In addition, difficulties arise in the case of the relative motion of the conductors, for example, in induction electromagnetic accelerators of solids or in the case of deformation of the conductors under the action of electromagnetic forces in the setups for producing ultrahigh magnetic fields under intense thermal and dynamic loading.

The boundary element (BE) method is used the most widely to calculate magnetic fields in nonconducting surrounding space [1–4]. This method does not require explicit calculations of the field outside the conductors, and, therefore, there is no need to construct and reconstruct (during motion and deformation of the conductors) a grid in the surrounding space, which reduces the computation time considerably. The finite element (FE) method proved suitable for calculating fields in conductors, in particular, in problems with complex geometry of conductors.

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The method combining these two methods and called the hybrid FE/BE method is currently widely used to solve electromagnetic problems [2–5]. One feature of the numerical modeling of the electromagnetic field produced by a system of conductors is the need to specify electrical voltages on these conductors. As a rule, these voltages are unknown and should be calculated during the solution or from a specified dependence of the total current through the conductors, or from an analysis of the operation of the setup taking into account the external electrical circuits providing energy supply. In both cases, as a rule, in each time step one uses an additional iterative cycle with the conductor voltages chosen so as to match the currents in these conductors with the specified currents in the external electrical circuits [2, 3] or with the currents and their time derivatives [5].

The present paper proposes a hybrid algorithm for calculating fields in systems of moving conductors taking into account external electrical connections without intermediate iterations.

1. Formulation of the Problem. We assume that the examined system of conductors is symmetric about the z axis. In such systems, it is convenient to study the distributions of electromagnetic fields and currents using cylindrical coordinates (r, θ, z) . We consider systems with azimuthal currents $\mathbf{j} = \mathbf{e}_\theta j$ which produce a poloidal magnetic field with induction $\mathbf{B} = \mathbf{e}_r B_r + \mathbf{e}_z B_z$. The components of the magnetic fields and currents depend on the spatial coordinates r and z and time t . Assuming that $\mathbf{B} = \nabla \times \mathbf{A}$, the magnetic potential vector is $\mathbf{A} = \mathbf{e}_\theta A_\theta$, and the magnetic induction vector components are given by the relations $B_r = -(1/r) \partial\psi/\partial z$ and $B_z = -(1/r) \partial\psi/\partial r$ ($\psi = rA_\theta$ is the stream function). Using the standard assumptions made in the analysis of high-current pulse devices in a quasistationary approximation (i.e., ignoring bias currents and setting the magnetic permeability of the medium $\mu = 1$), from the Maxwell equations for the stream function in the conductors we obtain the equation [2]

$$\frac{\sigma}{r} \frac{d\psi}{dt} + j_\theta = \frac{\sigma U_{\text{ext}}}{2\pi r}, \quad (1)$$

where $j_\theta = (1/\mu_0) \nabla(1/r) \nabla\psi$, U_{ext} is the electrical voltage produced by the external sources on the conductors, σ is the electrical conductivity, μ_0 is the magnetic permeability of vacuum, and $\nabla = \mathbf{e}_r (\partial/\partial r) + \mathbf{e}_z (\partial/\partial z)$ is a two-dimensional differential operator in the space (r, z) .

Because outside the conductors, $j_\theta = 0$, instead of (1) we have

$$\nabla \frac{1}{r} \nabla\psi = 0. \quad (2)$$

We assume that, generally, the examined system of conductors is represented in the plane (r, z) by a region $\Omega_c = \sum \Omega_k$ with the boundary Γ_c consisting of n_c simply connected subregions Ω_k ($k = 1, 2, \dots, n_c$) which have boundaries Γ_k . The nonconducting space surrounding the conductors occupies a region Ω_0 , which, in addition to the boundaries with the conductors, has an infinitely distant boundary Γ_∞ and a boundary Γ_0 which coincides with the z axis. On the boundaries of the conductors, the continuity conditions for the tangential and normal components of the magnetic induction vector should be satisfied. On the external infinitely distant boundary, the conditions $\psi|_{\Gamma_\infty} = 0$, $\nabla\psi|_{\Gamma_\infty} = 0$ and $(\partial\psi/\partial r)|_{\Gamma_0} = 0$ are satisfied. On the symmetry axis of the system (the boundary Γ_0), $B_r = 0$ and, hence, $(\partial\psi/\partial z)|_{\Gamma_0} = 0$; in addition, we can also set $\psi|_{\Gamma_0} = 0$. At the initial time, there are no currents in the conductors, and, hence, $\psi|_{t=0} = 0$. If the time dependences of the external voltages applied to the conductors $U_{\text{ext}}(t)$ are known, the problem can be considered to be formulated. In many cases, however, these voltage are unknown and should be calculated during the solution of the problem taking into account the parameters of the external electrical circuits and the connection layout.

2. Numerical Solution Procedure. We discretized the region of the conductors by a finite-element grid. The spatial distribution of the stream function and its time derivatives are approximated using functions of the form $N_i(r, z)$ ($i = 1, 2, \dots, n_p$) and the nodal values of the corresponding quantities ψ_i and $d\psi_i/dt$:

$$\psi \approx \sum_i N_i \psi_i, \quad \frac{d\psi}{dt} \approx \sum_i N_i \frac{d\psi_i}{dt}. \quad (3)$$

In the region of the conductors, Eq. (1) is solved using the Galerkin weighted residual finite-element method. As a result, we obtain a system of linear differential equations, which is written in matrix form as

$$M \frac{d\psi}{dt} + K\psi = \mathbf{Y}^t U + G \frac{\partial\psi}{\partial n}. \quad (4)$$

Here M , K , G , and U are matrices with elements

$$M_{ij} = \iint_{\Omega_c} \sigma \frac{N_i N_j}{r} ds, \quad K_{ij} = \iint_{\Omega_c} \frac{\nabla N_i \cdot \nabla N_j}{\mu r} ds, \quad G_{ij} = \sum_{k=1}^{n_c} \int_{\Gamma_k} \frac{N_i N_j}{r} dl, \quad U_{ij} = \delta_{ij} U_{k(i)};$$

δ_{ij} is the Kronecker symbol, \mathbf{Y} is a vector with components $Y_i = \iint_{\Omega} \sigma \frac{N_i}{2\pi r} ds$, and $\boldsymbol{\psi}$ is the vector of the nodal values of the stream function. The right side of Eq. (4) contains unknown nodal values of the normal derivatives of the stream function $\partial\psi/\partial n$, which should be determined from the continuity conditions for these derivatives on the boundary of the conductors during matching of the solutions of Eqs. (1) and (2) for the conductors and the nonconducting space around them.

Applying the weighted residual method to Eq. (2) with the weight function being the fundamental solution of (2) and using Green formula, we obtain the integral equation [1]

$$\frac{\beta_i}{2\pi} \psi_i + \sum_{k=1}^{n_c} \oint_{\Gamma_k} \frac{1}{r} \left(\psi \frac{\partial \psi_i^*}{\partial n} - \psi_i^* \frac{\partial \psi}{\partial n} \right) d\Gamma = 0 \quad (5)$$

(ψ^* is the fundamental solution). For the case considered [6], we have

$$\psi_i^* = \frac{1}{\pi} \sqrt{\frac{r_i r}{m_i}} \left(\left(1 - \frac{m_i}{2} \right) K(m_i) - E(m_i) \right),$$

where K and E are complete elliptic integrals of the first and second kind, $m_i = 4r_i r / ((r_i + r)^2 + (z_i - z)^2)$, the coefficient β_i is equal to the external angle of the boundary at the boundary node r_i, z_i .

The boundary $\partial\Omega_c$ is discretized into boundary elements Γ^b ($b = 1, 2, \dots, n_b$) so that they coincide with the edges of the finite-element grid. Similarly to (3), we interpolate the stream function and its normal derivative on the boundaries and calculate (5) at each boundary node. As a result, we obtain

$$H\boldsymbol{\psi} = Q \frac{\partial \boldsymbol{\psi}}{\partial n}, \quad (6)$$

where H and Q are matrices with elements

$$H_{ij} = \frac{\delta_{ij} \beta_i}{2\pi} + \sum_{k=1}^{n_c} \oint_{\Gamma_k} \frac{N_j}{r} \frac{\partial \psi_i^*}{\partial n} d\Gamma, \quad Q_{ij} = \sum_{k=1}^{n_c} \oint_{\Gamma_k} \frac{N_j}{r} \psi_i^* d\Gamma.$$

From the system of linear equations (6), we find the values of the normal derivatives of the stream function at the nodes on the boundaries of the conductors

$$\left. \frac{\partial \boldsymbol{\psi}}{\partial n} \right|_{\Gamma_c} = -Q^{-1} H \boldsymbol{\psi} \Big|_{\Gamma_c}$$

(the minus sign indicates that the direction of the external normal for the conducting subregions is opposite to the direction of the normal on the external boundaries of the nonconducting space). Substitution of this expression into (4) yields the complete system of ordinary differential equations for the nodal values of the stream function:

$$M \frac{d\boldsymbol{\psi}}{dt} + (K + K') \boldsymbol{\psi} = \mathbf{Y}^t U, \quad K' = GQ^{-1}H. \quad (7)$$

If the time dependences of the external voltages on the conductors $U_k(t)$ ($k = 1, 2, \dots, n_c$) are known, integration of system (7) with zero initial conditions yields values of the stream function at the subsequent times.

We obtain expressions for the total currents in the conductors. For the k th conductor, we have

$$I_k = \mu_0^{-1} \oint_{\Gamma_k} \mathbf{B} \cdot d\mathbf{l}$$

($d\mathbf{l}$ is a vector tangential to the boundary Γ_k). Taking into account that

$$\mathbf{B} \cdot d\mathbf{l} = -\frac{1}{r} \frac{\partial \boldsymbol{\psi}}{\partial n} dl$$

and using approximation (3), we have

$$\begin{aligned}
I_k &= \mu_0^{-1} \oint_{\Gamma_k} \mathbf{B} \cdot d\mathbf{l} = \sum_{b(\Gamma_k)} \int_{\Gamma^b} \mathbf{B} \cdot d\mathbf{l} \\
&\approx -\mu_0^{-1} \sum_{i(\Gamma_c)} \sum_{b(i)} \int_{\Gamma_b} N_i \frac{1}{\mu_0 r} \frac{\partial \psi}{\partial n} dl = \sum_{i(\Gamma_k)} \sum_{j(\Gamma_c)} K'_{ij} \psi_j = \mathbf{J}_k^t \psi
\end{aligned} \tag{8}$$

$(\mathbf{J}_k^t = \sum_{i(\Gamma_k)} K'_{ij})$ is the row vector. The expression $i(\Gamma_k)$ denotes the set of indices of the nodes belonging to the boundary Γ_k , and $b(i)$ denotes the set of indices of the boundary elements containing the node i , etc. We note that, in the k th conductor, the total current is determined by the values of the stream function at the nodes located on the boundaries of all conductors.

We consider the case where the time dependence of the total current $I_k(t)$ is known for the k th conductor. For simplicity, we assume that, on the remaining conductors of the system, the external voltage is equal to zero. Substituting the approximation of the stream function (3) into relation (1), integrating the result over the area of the k th conductor S_k , and taking into account that

$$\iint_{S_k} \frac{\sigma}{r} \frac{d\psi}{dt} ds = \sum_{i(S_k)} \frac{d\psi_i}{dt} \iint_{S_k} \frac{\sigma N_i}{r} ds = 2\pi \mathbf{Y}_k^t \frac{d\psi}{dt},$$

we have

$$U_k = 2\pi R_{0k} \left(\mathbf{Y}_k^t \frac{d\psi}{dt} + I_k \right), \tag{9}$$

where R_{0k} is the electrical resistance of the conductor for direct current:

$$R_{0k} = \left(\iint_{S_c} \frac{\sigma}{2\pi r} ds \right)^{-1}.$$

Substitution of the expression for the voltage (9) into system (7) yields

$$(M - 2\pi R_{0k} \mathbf{Y}_k \mathbf{Y}_k^t) \dot{\psi} + (K + K') \psi = \mathbf{Y}_k R_{0k} I_k(t).$$

Since the vector \mathbf{Y}_k has the nonzero components corresponding to the set of indices $i(\Omega_k)$ of the nodes in the conductor, the matrix M is added by a filled block $2\pi R_{0k} \mathbf{Y}_k \mathbf{Y}_k^t$, whose nonzero elements are on the intersection of the row and columns with the numbers belonging to the set of indices $i(\Omega_k)$.

It should be noted that this approach can also be used in the case $I_k(t) = 0$, which corresponds to an open conductor (a ring conductor with a thin radial cut). Such a situation arises in a system containing magnetic flux concentrators and (or) external conducting electrically open fasteners holding any of the conductors.

3. Coupling with an External Electrotechnical Circuit. Relations (8) and (9) can be used to calculate the voltage on the conductors connected with both external electrotechnical lumped-parameter circuits and with each other. We consider two typical examples.

3.1. R -, L -, and C -External Circuit. Let the k th conductor be connected to a capacitor bank with capacitance C by means of a transmission line with resistance R and inductance L , and let the external voltage on the remaining conductors be equal to zero. In this case, $U_k = U_C - RI_k - LI\dot{I}_k$ (U_C is the capacitor voltage). Substituting this expression into (7) and using the relations for the total current in the conductor (8), we obtain

$$(M + L\mathbf{Y}_k \mathbf{J}_k^t) \frac{d\psi}{dt} + \left(K + K' + R\mathbf{Y}_k \mathbf{J}_k^t + L\mathbf{Y}_k \frac{d\mathbf{J}_k^t}{dt} \right) \psi = U_C \mathbf{Y}_k. \tag{10}$$

To determine the capacitor voltage, it is necessary to supplement system (10) by the equation

$$\dot{U}_C = -\frac{I_k}{C} = -\frac{\mathbf{J}_k^t \psi}{C}, \quad U_C(0) = U_0.$$

The matrix $L\mathbf{Y}_k d\mathbf{J}_k^t/dt$ appears for any relative motion of the conductors or their deformations.

3.2. *Series Connection of Conductors.* We consider the case of series connection of conductors, which is encountered, for example, in calculations of fields produced by a multi-turn solenoid. Let m conductors be connected in series; in this case, each conductor carries an identical total current $I = I_1 = I_2 = \dots = I_k = \dots = I_m$. Let this group of conductors be subjected to a total external voltage

$$U_{\text{ext}} = \sum_{k=1}^m U_k. \quad (11)$$

We note that even in the case of conductors of identical geometry, the voltage drop across each of them is different. Using the equal current condition and relations (9), (11), we find the voltage on the conductors:

$$U_k = U_{\text{ext}} - \sum_{j \neq k} R_{0j} \left(\mathbf{J}_j^t \boldsymbol{\psi} + 2\pi \mathbf{Y}_j^t \frac{d\boldsymbol{\psi}}{dt} \right). \quad (12)$$

The voltages calculated by formula (12) can be used to calculate the matrix U in relations (7). If U_{ext} is the voltage transmitted to the conductors from the capacitor bank, the matrices M and K should be modified according to the procedure described in Sec. 3.1.

The proposed methods for taking into account the external connections of conductors lead to the filling of the mass matrix in the system of ordinary differential equations obtained. Thus, if a conductor is coupled to an external circuit with inductance (10), nonzero elements appear at the intersection of the rows with the numbers belonging to the set $i(\Omega_k)$ and at the intersection of the columns with the numbers belonging to the set of indices of the nodes located on the boundaries of all conductors $i(\Gamma_c)$. Accounting for the series connection of the conductors leads to the filling of the block in the mass matrix that corresponds to the indices of all nodes in series connected conductors. The matrix M does not depend on time if the electrical conductivity of the material is constant and the conductors are stationary during the calculation. In the case of motion of conductors and (or) accounting for electrical conductivity variation during their heating, the mass matrix depends on time. However, if the electrical conductivity can be considered constant in the problem studied, effective factorization of the mass matrix can also be performed for a system of moving conductors since, in the cases considered, the time-dependent additive can be represented as the matrix product of the column and row vectors.

4. Example of Numerical Modeling of a Multi-Turn Induction Accelerator of Metal Plates.

The algorithm presented here was used to design a computer program and model the operation of an induction accelerator with a multi-turn flat inductor for acceleration of disk-shaped metal bodies. This problem is similar to that considered in [7], where the inductor was modeled by a system of ring concentric infinitely thin conductors which carried identical specified currents. The magnetic field was calculated using the Biot–Savart law, but this approach does not provide information on the thermal load on the inductor and the efficiency of the accelerator. In the present work, we performed a complete modeling of the accelerator with a detailed calculation of the field and current distributions in both the accelerated body and the inductor. The total current in the system was determined with the real parameters of the external circuit taken into account.

The inductor was modeled by a system of concentric ring copper conductors with a cross section of 9×5 mm and a turn-to-turn distance of 0.5 mm. The internal radius of the inductor was 25 mm. The accelerated body of aluminum had a diameter of 68 mm, equal to the external diameter of the inductor, a thickness of 5 mm, and a mass of 200 g. The initial distance from the accelerated disk to the inductor was 0.5 mm. The finite-element grid used in the calculations (Fig. 1) included 3400 triangular elements, 1800 nodes, and 430 boundary elements. The velocity of the body and his displacement were determined by solving the equations of motion

$$M \frac{dV}{dt} = 2\pi \iiint_{\Omega_a} j_{\theta} B_r r \, dr \, dz, \quad \frac{dZ}{dt} = V.$$

It was assumed that the inductor is connected to a 1000 μF capacitor bank and the transmission line had a resistance of 1 m Ω and an inductance of 0.1 μH . The calculations were performed for an initial capacitor voltage of 10 kV. The voltage across the turns was calculated using (9). The difference in the total current between the inductor turns did not exceed 1%. The obtained time dependences of the total currents in the inductor and the accelerated body are presented in Fig. 2. Figure 3 gives time dependences of the velocity of the body and the distance traveled by the body. The system provides acceleration of the disk to a velocity of 450 m/sec. At the

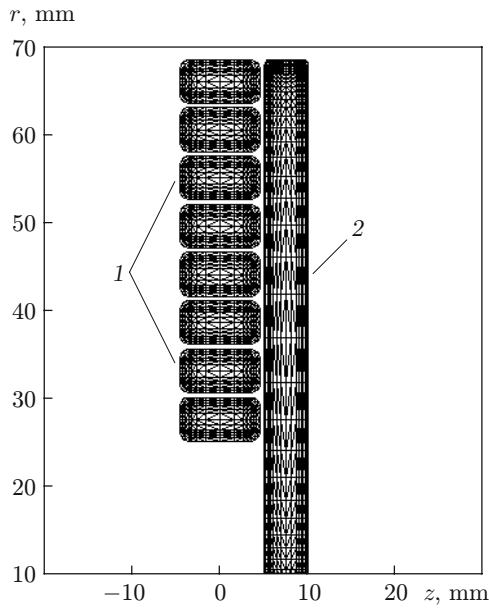


Fig. 1

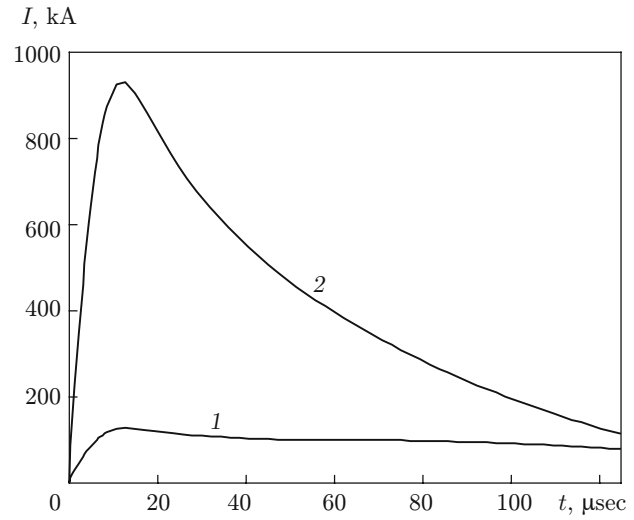


Fig. 2

Fig. 1. Diagrams of the arrangement of the finite-element grid for the inductor (1) and the accelerated body (2).

Fig. 2. Total currents in the inductor (1) and the accelerated body (2) versus time.

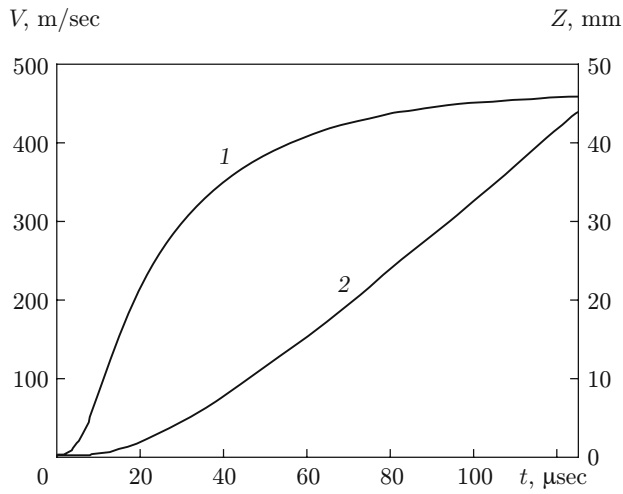


Fig. 3

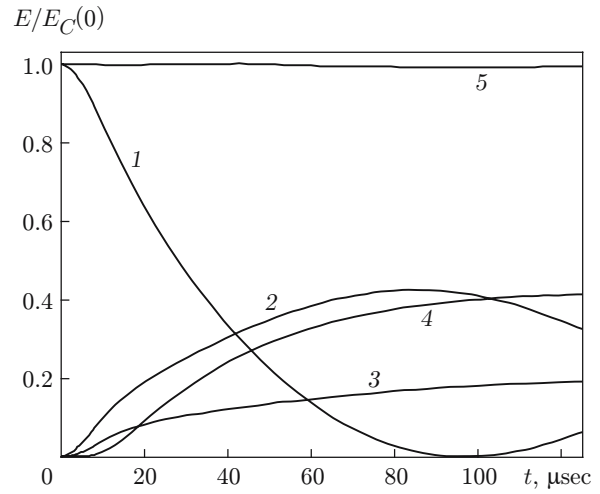


Fig. 4

Fig. 3. Velocity of the body (1) and the distance traveled by the body (2) versus time.

Fig. 4. Normalized time dependences of various kinds of energy: 1) capacitor energy; 2) magnetic-field energy; 3) Joule loss energy; 4) kinetic energy of the body; 5) sum of the energies.

end of the acceleration process, the maximum temperature of the inductor was 370°C, and that of the disk was 240°C. The maximum heating of the disk occurs in the zone corresponding to the average radius of the inductor. The efficiency of conversion of the electrical energy of the capacitor to kinetic energy is approximately 40%. For multi-turn systems, effective acceleration is achieved if the disk is displaced by a distance approximately equal to the double height of the inductor turn. Figure 4 shows the dynamics of various kinds of energy in the system (capacitor energy, magnetic-field energy of the system, Joule loss energy, and the kinetic energy of the body). It is evident that the sum of all kinds of energy differs from the initial capacitor energy by not more than 0.5%. Hence, the technique presented here is correct.

Conclusion. The proposed method for calculating nonstationary distributions of electromagnetic fields and currents in systems of inductively coupled moving conductors with external circuits taken into account provides complete modeling of the operation of various devices (for example, induction accelerators of solids with single- and multi-turn inductors) and electromagnetic field generating systems, including magnetic flux concentrators, conducting open holders of conductors, etc. Despite the significant filling of the mass matrix in the obtained system of ordinary differential equations, the algorithm does not require intermediate iterations and is fairly effective from a point of view of computation time.

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